

Optimal Spectra of Projective Definability

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Introduction

The topic of this talk is at the border between descriptive set theory, forcing, and combinatorial set theory.

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The topic of this talk is at the border between descriptive set theory, forcing, and combinatorial set theory.

- ▶ (Gödel) $V = L$ implies there exists a Δ_2^1 well-order of the reals.

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- ▶ (Gödel) $V = L$ implies there exists a Δ_2^1 well-order of the reals.
- ▶ (Mansfield) If there exists a Σ_2^1 well-order of the reals, then every real is constructible.

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- ▶ (Mansfield) If there exists a Σ_2^1 well-order of the reals, then every real is constructible.
- ▶ (Harrington, 1977) It is consistent with $\mathfrak{c} \geq \aleph_2$ that there exists a Δ_3^1 well-order of the reals.

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- ▶ (Harrington, 1977) It is consistent with $\mathfrak{c} \geq \aleph_2$ that there exists a Δ_3^1 well-order of the reals.

Theorem (S.D. Friedman, V. Fischer, 2010 ; [FF10])

It is consistent that $\mathfrak{b} < \mathfrak{a} = \mathfrak{s}$ and there exists a Δ_3^1 well-order of the reals.

Background and Motivation

Regarding maximal almost disjoint families $\mathcal{A} \subseteq [\omega]^\omega$, one may ask about the possible cardinality of $\mathcal{A}$, leading to the study of the set

$$\text{spec}(\mathfrak{a}) := \{\kappa \mid \kappa = |\mathcal{A}| \wedge \mathcal{A} \subseteq [\omega]^\omega \text{ is an infinite mad family}\}.$$

Classically the focus is on the cardinal

$$\mathfrak{a} := \min(\text{spec}(\mathfrak{a})).$$

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Controlling the spectrum

- ▶ Hechler showed in 1972 that it is consistent $2^{\aleph_0}$ is large and $\text{spec}(\mathfrak{a}) = [\omega_1, 2^{\aleph_0}]$.
- ▶ Blass specifies a sufficient set of conditions on a set $\Theta \subseteq [\omega_1, 2^{\aleph_0}]$ guaranteeing that there is a ccc poset $\mathbb{P}$ forcing $\Theta = \text{spec}(\mathfrak{a})$.
- ▶ Shelah and Spinas showed some of those assumptions can be removed.

Background and Motivation

One the other hand, one can also ask about the possible definability of a mad family as a subset of $[\omega]^\omega$:

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Definability of mad families

- ▶ (Mathias, 1969 ; [Mat77]): no infinite mad family can be analytic.

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- ▶ (Mathias, 1969 ; [Mat77]): no infinite mad family can be analytic.
- ▶ (Miller, 1984): $V = L$ implies there exists a Π_1^1 mad family.

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- ▶ (Mathias, 1969 ; [Mat77]): no infinite mad family can be analytic.
- ▶ (Miller, 1984): $V = L$ implies there exists a Π_1^1 mad family.
- ▶ (Törnquist, 2013; [Tö13]) The existence of a Σ_2^1 mad family is equivalent the existence of a Π_1^1 mad family.

In other words, in a model of $V = L$, we have that $\{\mathfrak{a}\} = \text{spec}(\mathfrak{a})$ has an optimal projective witness.

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Question

Under what conditions $\kappa \in \text{spec}(\mathfrak{a})$ have an optimal projective witness?

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Under what conditions $\kappa \in \text{spec}(\mathfrak{a})$ have an optimal projective witness?

Results

- $\aleph_1 = \mathfrak{a} < \mathfrak{c}$: by the existence of $\mathbb{P}$ -indestructible mad families in L , where $\mathbb{P}$ is a cardinal preserving forcing adding reals.

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- ▶ $\aleph_1 = \mathfrak{a} < \mathfrak{c}$: by the existence of $\mathbb{P}$ -indestructible mad families in L , where $\mathbb{P}$ is a cardinal preserving forcing adding reals.
- ▶ $\aleph_1 < \mathfrak{b} = \mathfrak{c} = \aleph_2$: consistent with a Π_2^1 tight, or ω -mad family (S.D. Friedman, L. Zdomskyy, 2010 ; [FZ10]).

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- ▶ $\aleph_1 < \mathfrak{b} = \mathfrak{c} = \aleph_2$: consistent with a Π_2^1 tight, or ω -mad family (S.D. Friedman, L. Zdomskyy, 2010 ; [FZ10]).
- ▶ $\mathfrak{b} = \mathfrak{c} = \kappa$ for any regular uncountable κ : consistent with a Π_1^1 mad family (J. Brendle, Y. Khomskii, 2013 ; [BK13]).

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- ▶ $\aleph_1 < \mathfrak{b} = \mathfrak{c} = \aleph_2$: consistent with a Π_2^1 tight, or ω -mad family (S.D. Friedman, L. Zdomskyy, 2010 ; [FZ10]).
- ▶ $\mathfrak{b} = \mathfrak{c} = \kappa$ for any regular uncountable κ : consistent with a Π_1^1 mad family (J. Brendle, Y. Khomskii, 2013 ; [BK13]).
- ▶ $\aleph_1 < \mathfrak{a} < \mathfrak{c} = \aleph_n$ for some $n \geq 3$: consistent with a Π_2^1 mad family (V. Fischer, S.D. Friedman, A. Törnquist, D. Schritterser, 2022; [FFST22]).

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Optimal Π_2^1 -definitions

- ▶ It is consistent $\aleph_1 < \mathfrak{a} < \mathfrak{c}$ and there is a Π_2^1 witness to $\mathfrak{a}$ (Fischer, Friedman, Schritterser, Törnquist; [FFST22]);
- ▶ The existence of a Π_2^1 -tight mad family is consistent with $\mathfrak{b} = \mathfrak{c} = \aleph_2$ (Friedman, Zdomskyy; [FZ10]).

These results are optimal:

- ▶ Mansfield's theorem states any Σ_2^1 -set is contained in L or contains a perfect subset of nonconstructible reals.
- ▶ Raghavan ([Rag09]) showed that no tight mad family can contain a perfect subset.

Projective well-orders

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A line of research running parallel to these results is the study of projective well-orders in the presence of nontrivial cardinal characteristic values.

Results

It has been shown that a Δ_3^1 well-order is consistent with the following :

- $\mathfrak{b} = \mathfrak{c} = \aleph_3$ and a Π_2^1 tight mad family (Fischer, Friedman, Zdomskyy, 2011; [FFZ11]).
- $\mathfrak{b} = \mathfrak{c} = \aleph_3$ and a Π_1^1 mad family (Fischer, Friedman, Khomskii, 2013; [FFK13]).
- $\aleph_2 = \mathfrak{p} = \mathfrak{b} < \mathfrak{a} = \mathfrak{s} = \mathfrak{c} = \aleph_3$ (V. Fischer, S.D. Friedman, L. Zdomskyy, 2013; [FFZ13]).
- $\aleph_1 = \mathfrak{a} < \mathfrak{c} = \aleph_2$ that there exists a Π_1^1 tight mad family of size $\aleph_1$ (J. Bergfalk, V. Fischer, C. Switzer 2022; [?]).

What's left

So far, the focus has been on finding optimal witnesses to one value of $\text{spec}(\alpha)$, namely its minimum.

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What's left

So far, the focus has been on finding optimal witnesses to one value of $\text{spec}(\mathfrak{a})$, namely its minimum. What remains open:

Question

How to control $\text{spec}(\mathfrak{a})$ while guaranteeing multiple values $\kappa \in \text{spec}(\mathfrak{a})$ have optimal witnesses ?

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Question

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An Approach to realizing $\text{spec}(\alpha) = \{\aleph_1, \aleph_2\}$ with optimal witnesses

- ▶ Fix a strong Π_1^1 witness to α in L , “strong” in the sense that it remains maximal in a large class of forcing extensions;
- ▶ Generically add an optimally definable mad family of size $\aleph_2$, ensuring the $\aleph_1$ sized mad family is preserved; Shoenfield absoluteness preserves the definability.

Tight mad families

The notion of “strong” we use is the notion of a tight, or ω -mad family:

Definition (Malykhin, [Mal89])

An almost disjoint family $\mathcal{A} \subseteq [\omega]^\omega$ is *tight* if for all $\{X_i \mid i \in \omega\} \subseteq \mathcal{I}(\mathcal{A})^+$, there exists $B \in \mathcal{I}(\mathcal{A})$ such that $|B \cap X_i| = \infty$ for all $i \in \omega$.

Where $\mathcal{I}(\mathcal{A})$ denotes the ideal on ω generated by $\mathcal{A}$, and $\mathcal{I}(\mathcal{A})^+$ is the dual filter.

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Where $\mathcal{I}(\mathcal{A})$ denotes the ideal on ω generated by $\mathcal{A}$, and $\mathcal{I}(\mathcal{A})^+$ is the dual filter.

- ▶ Kurilic in 2001 ([Kur01]) showed tight mad families are Cohen-indestructible.
- ▶ Guzmàn, Hrušák, and Tézlez (2020, [GHT20]) isolated an iterable property of proper forcing notions sufficient for guaranteeing the preservation of tight mad families.

Preserving tight mad families

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Definition (Guzmán, Hrušák, Téllez; [GHT20])

A proper notion $\mathbb{P}$ *strongly preserves the tightness of* $\mathcal{A}$ if for all $p \in \mathbb{P}$, countable $M \prec H_\theta$ such that $\mathbb{P}, \mathcal{A}, p \in M$ and all $B \in \mathcal{I}(\mathcal{A})$ such that $|B \cap Y| = \omega$ for each $Y \in \mathcal{I}(\mathcal{A})^+ \cap M$, there exists $q \leq p$ such that q is $(M, \mathbb{P})$ -generic and

$$q \Vdash \forall \dot{Z} \in (\mathcal{I}(\mathcal{A})^+ \cap M[\dot{G}]) |\dot{Z} \cap B| = \omega.$$

Preserving tight mad families

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$$q \Vdash \forall \dot{Z} \in (\mathcal{I}(\mathcal{A})^+ \cap M[\dot{G}]) |\dot{Z} \cap B| = \omega.$$

Lemma (Guzmán, Hrušák, Téllez; [GHT20])

Countable support iterations of forcings which strongly preserve the tightness of $\mathcal{A}$ also preserve the tightness of $\mathcal{A}$.

Adding a generic Π_2^1 -tight mad family

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Theorem (Friedman, Zdomskyy; [FZ10])

It is consistent that $\mathfrak{b} = \mathfrak{a} = \aleph_2$ and there exists a Π_2^1 -definable tight mad family.

- ▶ Countable support iteration of S -proper posets $\mathbb{Q}_\alpha$, for $\alpha < \omega_2$.
 - ▶ For α limit, $\mathbb{Q}_\alpha = \mathbb{K}$ adds a generic real $a_\alpha \subseteq \omega$, so that in the end $\mathcal{A} = \bigcup_{\alpha < \omega_2} a_\alpha$ is a tight mad family; the Π_2^1 -definition is achieved by almost disjoint coding (à la Solovay-Jensen).
 - ▶ For α successor, $\mathbb{Q}_\alpha = \mathbb{D}$, Hechler forcing for adding a dominating real.

Realizing $\text{spec}(\mathfrak{a}) = \{\aleph_1, \aleph_2\}$, with optimal witnesses

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We were able to show that the poset $\mathbb{K}$ to add the generic tight mad family does not itself add a dominating real:

Lemma (Fischer, M.)

The Friedman-Zdomskyy poset $\mathbb{K}$ preserves tight mad families.

Realizing $\text{spec}(\mathfrak{a}) = \{\aleph_1, \aleph_2\}$, with optimal witnesses

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Lemma (Fischer, M.)

The Friedman-Zdomskyy poset $\mathbb{K}$ preserves tight mad families.

Corollary (Fischer, M.)

It is consistent with $\aleph_1 = \mathfrak{a} < \mathfrak{c} = \aleph_2$ and there exists a Π_1^1 -tight mad family of size $\aleph_1$ and a Π_2^1 -tight mad family of size $\aleph_2$.

So that each $\kappa \in \text{spec}(\mathfrak{a})$ has an optimally definable, tight witness.

A model with $\aleph_1 = \mathfrak{a} < \mathfrak{s} = \aleph_2$

Question

Can we also control other cardinal characteristic values, namely can we have $\mathfrak{a} < \mathfrak{s}$?

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A model with $\aleph_1 = \mathfrak{a} < \mathfrak{s} = \aleph_2$

Question

Can we also control other cardinal characteristic values, namely can we have $\mathfrak{a} < \mathfrak{s}$?

Recall:

Definition

- ▶ $\mathfrak{s}$ is the smallest cardinality of a set $\mathcal{A} \subseteq [\omega]^\omega$ such that for every $a \in [\omega]^\omega$ there exists $b \in \mathcal{A}$ such that $|a \cap b| = |b \setminus a| = \omega$.
- ▶ $\mathfrak{b}$ is the smallest cardinality of a $\mathcal{B} \subseteq \omega^\omega$ such that there is no $f \in \omega^\omega$ with $f <^* g$ for all $g \in \mathcal{B}$.

- ▶ $\mathfrak{s} < \mathfrak{b}$ holds in the Hechler model.

$\aleph_1 = \mathfrak{a} < \mathfrak{s} = \aleph_2$ and optimal witnesses

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Theorem (Shelah, 1984 [She84])

It is consistent with ZFC that $\aleph_1 = \mathfrak{b} < \mathfrak{s} = \aleph_2$.

$\aleph_1 = \mathfrak{a} < \mathfrak{s} = \aleph_2$ and optimal witnesses

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Theorem (Shelah, 1984 [She84])

It is consistent with ZFC that $\aleph_1 = \mathfrak{b} < \mathfrak{s} = \aleph_2$.

- ▶ ω_2 -length countable support iteration over a model of CH ;
- ▶ The iterands $\mathbb{Q}$ are proper and “almost ω^ω -bounding”, thus preserve the ground model reals as an unbounded family.

$\aleph_1 = \mathfrak{a} < \mathfrak{s} = \aleph_2$ and optimal witnesses

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Our main result was the following:

Lemma (Fischer, M.)

Shelah's poset $\mathbb{Q}$ strongly preserves ground model tight mad families.

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Our main result was the following:

Lemma (Fischer, M.)

Shelah's poset $\mathbb{Q}$ strongly preserves ground model tight mad families.

With careful bookkeeping and the fact that adding a Δ_3^1 well-order (as in [FF10]) preserves tight mad families, we get:

Theorem (Fischer, M.)

It is consistent with $\aleph_1 = \mathfrak{a} < \mathfrak{s} = \mathfrak{c} = \aleph_2$ that there exists a Δ_3^1 well-order of the reals, a Π_1^1 tight mad family of size $\aleph_1$, and a Π_2^1 tight mad family of size $\aleph_2$.

Conclusion

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This work can lead to the following questions:

- ▶ Can we demand $\mathfrak{c} \geq \aleph_3$?
- ▶ What other types of combinatorial families (such as eventually different families or maximal cofinitary groups) are preserved by Shelah's forcing?

It's over

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Thank you !

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